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ON THE MAXIMUM OF THE VERTICAL FALL VELOCITY OF A HOMOGENEOUS SPHERE WHOSE RADIUS CHANGES BY THE EXPONENTIAL LAW

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The extreme properties of the vertical fall of a spherical body whose radius decreases with time by the exponential law have been investigated. The conditions under which the fall velocity has a maximum have been found.

Keywords: *maximum of the fall velocity, vertical fall of a sphere, exponential law of decrease in the radius, hovering velocity, Bessel function, Lambert functions, velocity asymptotics.*

Introduction. In nature and in engineering, one can often see motion (flight) of variable-mass bodies, for example, rockets of various systems whose mass decreases in the process of fuel combustion. A change in the mass may also take place in the course of displacement of system elements (reeling up threads or ropes on drums), in evaporation of moving drops in a gaseous medium, in flight in the atmosphere of burning-out solid fuel particles, meteorites, etc.

Investigating the motion of a constant-mass solid body with account for the nonlinear drag of the gaseous medium, N. E. Zhukovskii [1] showed that the fall velocity of a the body is a monotonic function. Depending on the initial value, it either increases monotonically or decreases approaching asymptotically the limit called the hovering velocity. In the case of fall of a body whose mass decreases, a radical difference from the results of [1] is observed. This shows up first as the fact that the flight time is limited by not only the actual height of fall (stationary solid obstacle) but also by the lifetime (burning, evaporation) of the body as a moving object. Second, as the mass decreases, there is not a constant limiting value called the hovering velocity. Third, there exists a condition where the monotonicity of the change in the fall velocity is disturbed and it can have a maximum [2]. Therefore, below we formulate the problem of determining the conditions under which there appears an extremum in the fall velocity of a homogeneous sphere of variable radius and obtaining relations for calculations.

Formulation of the Problem. We shall conduct the investigation on the assumption that the aerodynamic drag force is proportional to the midsection area and the fall velocity of the sphere in the air squared. The sphere density is taken to be constant. Therefore, a decrease in its radius is accompanied by a decrease in its mass and an increase in the specific aerodynamic drag force.

Let us subject the change in the radius of the body to the exponential law

$$r(t) = r_0 \exp(-\lambda t) . \tag{1}$$

Note that the motion of the material point in which the mass changed exponentially was considered in [3, 4] and in other studies. Unlike the above publications, here we consider the flight of a body of variable sizes and mass.

Within the framework of the assumptions made, the change in the fall velocity of the body v with time t is described by the nonlinear differential equation with variable coefficients

$$\frac{dv}{dt} + \frac{k}{r} v^2 = g . \tag{2}$$

The initial fall velocity of the body is assumed to be equal to v_0. Therefore, the solution of Eq. (2) should satisfy the initial condition

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$$v(0) = v_0 . \tag{3}$$

Search of an Analytical Solution. To solve Eq. (2), let us introduce a new dimensionless variable

$$\xi = \exp(\lambda t), \quad \frac{d\xi}{dt} = \lambda\xi . \tag{4}$$

Taking into account that according to (4)

$$\frac{d}{dt} = \frac{d}{d\xi}\frac{d\xi}{dt} = \lambda\xi \frac{d}{d\xi},$$

we give (2) in the form

$$\frac{dv}{d\xi} + \beta v^2 = \frac{g_1}{\xi}, \tag{5}$$

where $g_1 = g/\lambda$; $\beta = k/(\lambda r_0)$.

To get rid of the nonlinearity in (5), we use the transformation

$$v = \frac{1}{\beta}\frac{dw}{d\xi} w^{-1}, \tag{6}$$

in which $w(\xi)$ is the unknown auxiliary function. Substituting expression (6) into (5), we obtain a Bessel-type linear equation

$$\frac{d^2 w}{d\xi^2} - \frac{\beta_1}{\xi} w = 0, \tag{7}$$

where $\beta_1 = \beta g_1$. The general solution of (7) is

$$w = \sqrt{\xi}\left(c_1 I_1(\eta) + c_2 K_1(\eta)\right), \tag{8}$$

where $\eta = 2\sqrt{\beta_1 \xi}$; c_1, c_2 are arbitrary constants.

Taking into account the above notation, after substituting (8) into (6) we obtain the general solution of Eq. (2)

$$v(\eta) = \frac{1}{\beta}\sqrt{\frac{\beta_1}{\xi}}\frac{cI_0(\eta) - K_0(\eta)}{cI_1(\eta) + K_1(\eta)}, \tag{9}$$

where $c = c_1 c_2^{-1}$ is an arbitrary constant. We determine the constant c in (9) from condition (3), which yields

$$c = \frac{v_0 \beta K_1(\eta_0) + \sqrt{\beta_1} K_0(\eta_0)}{\sqrt{\beta_1} I_0(\eta_0) - v_0 \beta I_1(\eta_0)}, \quad \eta_0 = 2\sqrt{\frac{\beta g}{\lambda}} . \tag{10}$$

Let us analyze the extreme properties of expression (9). From Eq. (2) it follows that the monotonicity of $v(t)$ is disturbed when

$$v_0 < \sqrt{\frac{g r_0}{k}} . \tag{11}$$

At the extreme point the derivative $dv/d\xi = 0$. Therefore, according to Eq. (5)

$$v_{max} = \sqrt{\frac{gr_0}{k\xi_e}} . \quad (12)$$

The value of ξ_e is the root of the transcendental equation

$$\frac{cI_0(\eta_e) - K_0(\eta_e)}{cI_1(\eta_e) + K_1(\eta_e)} = 1 , \quad (13)$$

in which $\eta_e = 2\sqrt{\beta_1 \xi_e}$. This root can be calculated with high accuracy by the Newton law. Assuming in the initial approximation $\xi_e = 1$, it is easy to find the subsequent values by the recurrent formula

$$\xi_{n+1} = \xi_n - \frac{\beta v^2 - \dfrac{g_1}{\xi_n}}{2\beta v\left(\beta v^2 - \dfrac{g_1}{\xi_n}\right) + \dfrac{g_1}{\xi_n^2}} , \quad (14)$$

where for calculating $v = v(\xi_n)$ solution (9) should be used.

Having determined with a given accuracy $\xi_e = \xi_{n+1}$, it is easy to determine the time of reaching the extremum since in accordance with (4)

$$t_e = \frac{1}{\lambda} \ln \xi_e . \quad (15)$$

An approximate determination of ξ_e can also be carried out by another method. To this end, let us reduce (13) to the form

$$c = \frac{K_1(\eta_e) + K_0(\eta_e)}{I_0(\eta_e) - I_1(\eta_e)} . \quad (16)$$

To simplify expression (14) at small η_e, let us use the asymptotics of small argument cylindrical functions [5]

$$I_0(\eta_e) \sim 1 , \quad I_1(\eta_e) \sim \frac{1}{2}\eta_e , \quad K_0(\eta_e) \sim \ln\frac{2}{\varepsilon\eta_e} , \quad K_1(\eta_e) \sim \frac{1}{\eta_e} , \quad (17)$$

where $\ln \varepsilon \approx 0.5772$ ... . Then at $\eta_e < 0.01$ instead of (16) we get $c \approx 1/\eta_e$.

Thus, if the initial parameters of motion are such that the value calculated by formula (10) is $c < 0.01$, then $\xi_e \approx (4c^2\beta_1)^{-1}$.

The other roots of Eq. (16) that lie in the interval $\eta_e \in [0.01; 5]$ can be determined approximately by means of a specially compiled table and linear interpolation, and Eq. (16) can also be simplified at large η_e. Using the asymptotics of large argument cylindrical functions [5]

$$I_0(\eta) - I_1(\eta) \sim \frac{\exp\eta}{2\eta\sqrt{2\pi\eta}} , \quad K_0\,\eta + K_1(\eta) \sim \frac{\sqrt{2\pi}}{\sqrt{\eta}} \exp(-\eta) ,$$

instead of (16) we get approximately

$$-2\eta_e \exp(-2\eta_e) = -\frac{c}{2\pi} . \quad (18)$$

The solution of Eq. (18) is expressed in terms of the known Lambert $W(z)$ function [6] in the form

TABLE 1. Average Roots of Eq. (16) on the Internal $\eta_e \in [0.01, 5]$

η_e	ln c	η_e	ln c	η_e	ln c	η_e	ln c
0.01	4.656	0.50	1.164	1.70	–0.577	3.40	–3.167
0.02	3.999	0.55	1.071	1.80	–0.716	3.50	–3.333
0.03	3.623	0.60	0.983	1.90	–0.857	3.60	–3.500
0.04	3.361	0.65	0.899	2.00	–0.999	3.70	–3.668
0.05	3.161	0.70	0.819	2.10	–1.143	3.80	–3.837
0.06	3.000	0.75	0.741	2.20	–1.288	3.90	–4.007
0.07	2.864	0.80	0.665	2.30	–1.436	4.00	–4.178
0.08	2.749	0.85	0.592	2.40	–1.585	4.10	–4.350
0.09	2.647	0.90	0.519	2.50	–1.736	4.20	–4.522
0.10	2.557	0.95	0.448	2.60	–1.889	4.30	–4.696
0.15	2.213	1.00	0.378	2.70	–2.043	4.40	–4.870
0.20	1.971	1.10	0.240	2.80	–2.200	4.50	–5.045
0.25	1.782	1.20	0.104	2.90	–2.357	4.60	–5.220
0.30	1.626	1.30	–0.032	3.00	–2.516	4.70	–5.396
0.35	1.492	1.40	–0.167	3.10	–2.677	4.80	–5.573
0.40	1.372	1.50	–0.303	3.20	–2.839	4.90	–5.750
0.45	1.264	1.60	–0.440	3.30	–3.002	5.00	–5.928

$$\eta_e = -\frac{1}{2} W\left(-\frac{c}{2\pi}\right). \tag{19}$$

Note that tables of Lambert functions are available in [7, 8] and can be used to determine η_e.

The accuracy of the asymptotic approximation (18) increases with increasing η_e. It follows from this approximation that at $\eta_e = 5$ ln $c \approx -5.86$ instead of ln $c \approx -5.93$ given in the table. So at $\eta_e > 5$ the root can be calculated by formula (19).

Calculation Results and Analysis. To evaluate the above theory, let us calculate the extremum parameters at $r_0 = 2\cdot 10^{-3}$ m; $k = 10^{-4}$; $\lambda = 0.5$ sec^{-1}; $v_0 = 5$ m/sec. Inequality (11) holds because $5 < \sqrt{gr_0/k} = 14.007$. This means that at such initial velocity there exists a maximum of the flying velocity. To solve the transcendental equation (13), we choose the initial approximation $\xi_e = 1$. The subsequent values are obtained by formulas (9), (10), (14): $\xi_2 = 1.466$, $\xi_3 = 1.710$, $\xi_4 = 1.832$, $\xi_5 = 1.860$, $\xi_6 = 1.861$, $\xi_7 = 1.861$. Six iterations have enabled us to determine with high accuracy the root of the transcendental equation. To this end, we find the values of the root by formulas (12) and (15): $v_{max} = 10.268$ m/sec, $t_e = 1.24$ sec.

For comparison, let us determine ξ_e by the second of the proposed techniques. For the assumed initial data ln $c = -3.874$, by the table we get $\eta_e = 3.80$, i.e., $\xi_e = 1.840$. Comparing the obtained ξ_e with ξ_7, we can see that the results are close.

Let us show how the initial velocity influences the behavior of the function $v(t)$. To this end, preserving the previous initial data, we give $v_0 = 0$, 7, 14.007, 20, and 25 m/sec. The calculation by formulas (9) and (10) for these values of v_0 leads, respectively, to curves 1–5 in Fig. 1. From this figure it is seen that curves 1 and 2 have a maximum. Curve 3 corresponds to the boundary value of $v_0 = \sqrt{gr_0/k}$, and curves 4 and 5 are monotonically decreasing. With increasing time all velocity-time curves approach one line, which is given analytically by the expression

$$v_a(t) = \sqrt{\frac{gr_0}{k}} \exp(-0.5\lambda t). \tag{20}$$

We arrive at this expression from solution (9) by replacing the cylindrical functions there by their asymptotic expressions at large argument values [5].

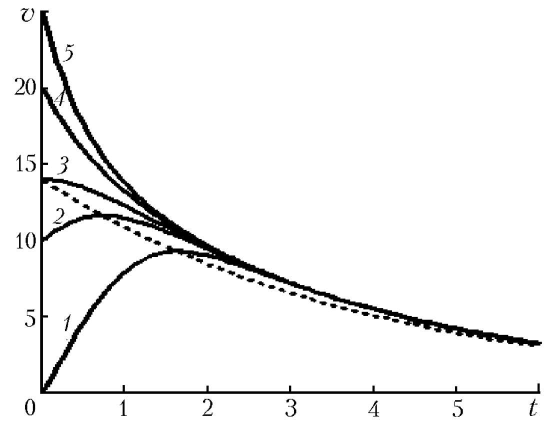

Fig. 1. Time dependence of velocity at equal values of v_0. v, m/sec; t, sec.

From formula (20) it follows that whatever the initial value of v_0, the velocity of fall of a sphere of decreasing radius at large t decreases with time by the exponential law. But the exponential laws of change in the radius and the velocity of the sphere have different exponents. The ratio of the exponent is equal to 2.

Assuming in formula (20) $\lambda = 0$ we get

$$v_a = \sqrt{\frac{gr_0}{k}} = \text{const},$$

which agrees with the value of the hovering velocity according to N. E. Zhukovskii [1] for a constant-mass body. Asymptotic analysis shows that a homogeneous sphere of decreasing dimensions has no constant hovering velocity. Its fall velocity decreases in the course of time. In the limit at $t \to \infty$ it equals zero but the object of motion itself, in which $r(\infty) = 0$, disappears therewith.

In Fig. 1, the dotted line marks asymptotics (20). It is seen from this figure that with increasing t independent of the initial velocity all the curves tend to the obtained asymptotics.

CONCLUSIONS

1. It has been established that the hovering velocity is absent as a homogeneous sphere with a decreasing radius is falling in a gas medium.
2. The existence conditions of a maximum of the fall velocity, as well as of its absence, have been obtained analytically.
3. The nonlinear character of the flying velocity asymptotics in the course of time has been determined.

NOTATION

g, gravitational acceleration; $I_n(z)$, $K_n(z)$, modified Bessel and Macdonald functions of order n; k, dimensionless aerodynamic drag coefficient; r, r_0, current and initial radii of the sphere; t, time; $v(t)$, v_0, current and initial vertical fall velocities of the body; $W(z)$, Lambert function; ε, Euler constant; λ, coefficient characterizing the rate of decrease in the sphere radius with time. Subscripts: a, asymptotic; 0, initial; e, extreme; max, maximum.

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